
Moduli Space of Self-Dual Gauge Fields, Holomorphic Bundles and Cohomology Sets¹

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Abstract. We discuss the twistor correspondence between complex vector bundles over a self-dual four-dimensional manifold and holomorphic bundles over its twistor space and describe the moduli space of self-dual Yang-Mills fields in terms of Čech and Dolbeault cohomology sets. The cohomological description provides the geometric interpretation of symmetries of the self-dual Yang-Mills equations.

1. Introduction

The purpose of this paper is to describe the moduli space of self-dual Yang-Mills fields and a symmetry algebra acting on the solution space of the self-dual Yang-Mills equations. The description of the moduli space of self-dual Yang-Mills fields is based on the twistor construction [10, 14, 1].

Let us briefly outline the differential-geometric background. We take M to be an oriented Riemannian 4-manifold, G a semi-simple Lie group, $P(M, G)$ a principal fibre bundle over M with the structure group G , A a connection 1-form on P , F_A its curvature 2-form and D a covariant differential on P . A connection 1-form A on P is called *self-dual* if its curvature F_A is *self-dual*, i.e.,

$$*F_A = F_A, \quad (1)$$

where $*$ is the Hodge star operator acting on 2-forms on M . We shall call eqs.(1) the *self-dual Yang-Mills* (SDYM) equations. By virtue of the Bianchi identity $D F_A = 0$, solutions of the SDYM equations automatically satisfy the Yang-Mills equations

$$D(*F_A) = 0. \quad (2)$$

Notice that solutions to eqs. (2) are of considerable physical importance (see the talk in this volume by S.T.Tsou [13]). Physicists use Yang-Mills theory (by which we mean any non-Abelian gauge theory) to describe the strong and electroweak interactions (see e.g. [3]). They call the connection 1-form A the *gauge potential* and the curvature 2-form F_A the *gauge* or *Yang-Mills field*. The SDYM equations (1) describe a subclass of solutions to the Yang-Mills equations (2). A choice of different boundary conditions for self-dual gauge fields gives such important solutions of the Yang-Mills equations as instantons, monopoles and vortices.

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It is well known that the SDYM equations are manifestly invariant under the gauge transformations of the gauge potential A and gauge field F_A and under the rescaling of a metric \mathbf{g} on M : $\mathbf{g} \mapsto e^\varphi \mathbf{g}$ (Weyl transformation), where φ is an arbitrary smooth function on M . The gauge transformations have the form (cf. [13])

$$A \mapsto A^g = g^{-1} A g + g^{-1} dg, \quad (3a)$$

$$F_A \mapsto F_A^g = g^{-1} F_A g, \quad (3b)$$

where g is a global section of the associated bundle of groups $\text{Int } P = P \times_G G$ (G acts on itself by internal automorphisms: $h_1 \mapsto h_2^{-1} h_1 h_2$, $h_1, h_2 \in G$), i.e., $g \in \Gamma(M, \text{Int } P)$. We shall denote the infinite-dimensional Lie group $\Gamma(M, \text{Int } P)$ by \mathfrak{G}_M and call it the *gauge group*.

Let us denote by \mathcal{A}_M the space of smooth global solutions to eqs. (1). The *moduli space* \mathcal{M} of self-dual gauge fields is the space of gauge nonequivalent self-dual gauge potentials on M ,

$$\mathcal{M} := \mathcal{A}_M / \mathfrak{G}_M. \quad (4)$$

Let $U \subset M$ be such an open ball that the bundle P is trivializable over U . We consider smooth self-dual connection 1-forms A on U , i.e., *local solutions* of the SDYM equations. Denote by \mathcal{A}_U the space of all smooth solutions to eqs.(1) on U and by \mathcal{M}_U the *moduli space* of smooth self-dual gauge potentials A on U ,

$$\mathcal{M}_U := \mathcal{A}_U / \mathfrak{G}_U, \quad (5)$$

where $\mathfrak{G}_U := \Gamma(U, \text{Int } P) = C^\infty(U, G)$ is an infinite-dimensional group of local gauge transformations.

The use of the moduli spaces (4) and (5) in physics is discussed in the talk by S.T.Tsou [13]. An important example of their use in mathematics is given by Donaldson's discovery of exotic smooth structures on 4-manifolds, which is based on topological properties of the moduli space of self-dual gauge fields over the manifolds in question [4].

The paper is organized as follows: in §2 we recall the twistor description of self-dual manifolds and self-dual gauge fields, in §3 we discuss the cohomological description of the moduli space of self-dual gauge fields mainly following the paper [12], and in §4 we describe the infinitesimal symmetries of the SDYM equations from the cohomological point of view (see also [6, 7]).

2. An important tool: Twistors

Twistors were introduced by Penrose in order to translate the massless free-field equations in space-time into holomorphic structures on a related complex manifold known as a twistor space. The twistor theory is based on an integro-geometric transformation which transforms complex-analytic data on the twistor space to solutions of massless field equations. Suggested originally for the description of linear conformally invariant equations, the twistor method has proved very fruitful for solving nonlinear equations of general relativity and Yang-Mills theories. Namely, the Penrose nonlinear graviton construction [10] gives the general local solution of the self-dual conformal gravity equations, and the Ward twistor interpretation of self-dual gauge fields [14] gives the general local solution of the SDYM equations on self-dual 4-manifolds M .

2.1. Twistor spaces

For each oriented Riemannian 4-manifold M one can introduce the manifold

$$\mathcal{Z} := P(M, SO(4))/U(2) \simeq P(M, SO(4)) \times_{SO(4)} S^2,$$

where $P(M, SO(4))$ is the principal $SO(4)$ -bundle of oriented orthogonal frames on M . So, the space \mathcal{Z} is a bundle associated to $P(M, SO(4))$ with the typical fibre $\mathbb{C}P^1 \simeq S^2$ and the canonical projection $\pi : \mathcal{Z} \rightarrow M$. The manifold \mathcal{Z} is called the *twistor space* of M .

A Riemannian metric \mathbf{g} is self-dual if the anti-self-dual part of the Weyl tensor vanishes [10, 1, 15]. Manifolds M with self-dual metrics are called *self-dual*. In [10, 1] it was shown that the twistor space \mathcal{Z} for such M is a complex 3-manifold. In what follows, we shall consider a self-dual manifold M and the twistor space \mathcal{Z} of M .

The Levi-Civita connection on M generates the splitting of the tangent bundle $T(\mathcal{Z})$ into a direct sum

$$T(\mathcal{Z}) = V \oplus H \quad (6)$$

of the vertical $V = \text{Ker } \pi_*$ and horizontal H distributions. The complexified tangent bundle of \mathcal{Z} can be splitted into a direct sum

$$T^{\mathbb{C}}(\mathcal{Z}) = V^{\mathbb{C}} \oplus H^{\mathbb{C}} = T^{1,0} \oplus T^{0,1} \quad (7)$$

of subbundles of type $(1,0)$ and $(0,1)$. Analogously one can split the complexified cotangent bundle of \mathcal{Z} into a direct sum of subbundles $T_{1,0}$ and $T_{0,1}$. Using the standard complex structure on $S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathcal{Z}$, one obtains

$$T^{\mathbb{C}}(\mathcal{Z}) = (V^{1,0} \oplus H^{1,0}) \oplus (V^{0,1} \oplus H^{0,1}). \quad (8)$$

The distribution $V^{0,1}$ is integrable.

Denote by $\{V_a\}$, $\{\bar{V}_a\}$, $\{\theta^a\}$ and $\{\bar{\theta}^a\}$ ($a = 1, 2, 3$) local frames for the bundles $T^{1,0}$, $T^{0,1}$, $T_{1,0}$ and $T_{0,1}$, respectively. Because of (8), each of the local frames is spanned by horizontal (when $a = 1, 2$) and vertical (when $a = 3$) parts. The derivative operator d on \mathcal{Z} splits as follows:

$$d = \partial + \bar{\partial}, \quad \partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0, \quad (9)$$

where locally $\partial = \theta^a V_a$, $\bar{\partial} = \bar{\theta}^a \bar{V}_a$.

Let us consider a sufficiently small open ball $U \subset M$ such that $\mathcal{Z}|_U$ is a direct product $\mathcal{P} \equiv \mathcal{Z}|_U \simeq U \times S^2$ as a smooth real 6-manifold. The space $\mathcal{P} \subset \mathcal{Z}$ is called the *twistor space* of U . This space is covered by two coordinate patches \mathcal{U}_1 and \mathcal{U}_2 ,

$$\mathcal{U}_1 := U \times \Omega_1, \quad \mathcal{U}_2 := U \times \Omega_2,$$

where $\Omega_1 = \{\lambda \in \mathbb{C} : |\lambda| < \infty\}$, $\Omega_2 = \{\zeta \in \mathbb{C} : |\zeta| < \infty\}$ form the covering $\Omega = \{\Omega_1, \Omega_2\}$ of the complex projective line $\mathbb{C}P^1$ and $\lambda = \zeta^{-1}$ on $\Omega_1 \cap \Omega_2$. On \mathcal{U}_1 and \mathcal{U}_2 we have the local coordinates $\{x^\mu, \lambda, \bar{\lambda}\}$ and $\{x^\mu, \zeta, \bar{\zeta}\}$, respectively. We denote by $\mathfrak{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ the two-set open covering of $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ and by \mathcal{U}_{12} the intersection $\mathcal{U}_1 \cap \mathcal{U}_2 = U \times (\Omega_1 \cap \Omega_2)$.

Recall that for any self-dual manifold its twistor space is a complex manifold. So, on $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{P}$ one can introduce holomorphic coordinates $\{z_1^a\}$, $\{z_2^a\}$, $a = 1, 2, 3$. On the intersection $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$ these coordinates are related by a holomorphic transition function $f_{12} : z_1^a = f_{12}^a(z_2^b)$. For local frames $\{\bar{V}_a^{(1)}\}$ and $\{\bar{V}_a^{(2)}\}$ of the bundle $T^{0,1}$ over \mathcal{U}_1 and \mathcal{U}_2 one has $\bar{V}_a^{(1)} z_1^b = 0$ on \mathcal{U}_1 and $\bar{V}_a^{(2)} z_2^b = 0$ on \mathcal{U}_2 . Notice that as local frames of $T^{0,1}$ over $\mathcal{U}_1, \mathcal{U}_2$ one can take the antiholomorphic vector fields $\{\frac{\partial}{\partial \bar{z}_1^a}\}$ on \mathcal{U}_1 and $\{\frac{\partial}{\partial \bar{z}_2^a}\}$ on \mathcal{U}_2 .

2.2. Twistor correspondence

Let M be a self-dual 4-manifold with the twistor space \mathcal{Z} . There is a bijective correspondence [14, 2, 1] between complex vector bundles $E \rightarrow M$ on M with self-dual connections and holomorphic vector bundles $\tilde{E} \rightarrow \mathcal{Z}$ on \mathcal{Z} which are trivial on fibres \mathbb{CP}^1 of the bundle $\pi : \mathcal{Z} \rightarrow M$ (see also [11, 16, 8] and references therein).

Let us briefly describe the twistor correspondence for the case of a vector bundle \mathcal{E} over an open set $U \subset M$ with a self-dual connection 1-form A . Such a bundle (\mathcal{E}, A) can be lifted to a bundle $(\pi^*\mathcal{E}, \pi^*A)$ over the twistor space \mathcal{P} of U . By definition of the pull-back, the pulled back connection 1-form π^*A on $\pi^*\mathcal{E}$ is flat along the fibres \mathbb{CP}^1 of the bundle $\pi : \mathcal{P} \rightarrow U$. Therefore, the components of the connection 1-form π^*A on the bundle $\tilde{\mathcal{E}}_0 := \pi^*\mathcal{E}$ along the distribution V can be set equal to zero. Moreover, the bundle $\tilde{\mathcal{E}}_0$ is a trivial complex vector bundle $\tilde{\mathcal{E}}_0 = \mathcal{P} \times \mathbb{C}^n$ with the transition matrix $\mathcal{F}_{12}^0 = 1$ on $\mathcal{U}_1 \cap \mathcal{U}_2$. As it was demonstrated in [14, 2, 1], the SDYM equations (1) on a connection 1-form A on \mathcal{E} is the condition for the connection 1-form π^*A to define a *holomorphic structure* on the bundle $\tilde{\mathcal{E}}_0$. Namely, the 1-form π^*A can be splitted into a direct sum of $(1, 0)$ - and $(0, 1)$ -parts, and the operator $\bar{\partial}$ can be lifted from \mathcal{P} to $\tilde{\mathcal{E}}_0$,

$$\bar{\partial}_{\tilde{B}} = \bar{\partial} + \bar{B}, \quad (10)$$

where \bar{B} is the $(0, 1)$ -part of π^*A satisfying the equations

$$\bar{\partial}_{\tilde{B}}^2 \equiv \bar{\partial}\bar{B} + \bar{B} \wedge \bar{B} = 0. \quad (11)$$

In the local frame $\{\bar{\theta}^a\}$, $a = 1, 2, 3$, we have $\bar{B} = \bar{B}_a \bar{\theta}^a$ and $\bar{B}_3 = 0$. Let us denote the described correspondence as $(\mathcal{E}, A) \sim (\tilde{\mathcal{E}}_0, \bar{B})$. From (11) it follows that the trivial holomorphic vector bundle $\tilde{\mathcal{E}}_0$ with the flat $(0, 1)$ -connection \bar{B} is diffeomorphic to a *holomorphic vector bundle* $\tilde{\mathcal{E}}$ with a holomorphic transition matrix \mathcal{F}_{12} , i.e., $(\tilde{\mathcal{E}}_0, \bar{B}) \sim (\tilde{\mathcal{E}}, \mathcal{F}_{12})$. Therefore, there exist smooth G -valued functions ψ_1 on \mathcal{U}_1 and ψ_2 on \mathcal{U}_2 such that $\bar{B}_a^{(1)} = -(\bar{V}_a^{(1)}\psi_1)\psi_1^{-1}$, $\bar{B}_a^{(2)} = -(\bar{V}_a^{(2)}\psi_2)\psi_2^{-1}$ and $\mathcal{F}_{12} = \psi_1^{-1}\mathcal{F}_{12}^0\psi_2 = \psi_1^{-1}\psi_2$, where $\mathcal{F}_{12}^0 = 1$ is the transition matrix in the bundle $\tilde{\mathcal{E}}_0$. Since \bar{B} is zero along the distribution $V^{0,1}$, we have $\bar{V}_3^{(1)}\psi_1 = 0$ on \mathcal{U}_1 and $\bar{V}_3^{(2)}\psi_2 = 0$ on \mathcal{U}_2 , which means that $\tilde{\mathcal{E}}$ is holomorphically trivial after the restriction to any projective line $\mathbb{CP}_x^1 \hookrightarrow \mathcal{P}$, $x \in U$.

To sum up, we have a one-to-one correspondence between the complex vector bundle \mathcal{E} over $U \subset M$ with a self-dual connection 1-form A and the trivial complex vector bundle $\tilde{\mathcal{E}}_0$ over \mathcal{P} with the flat $(0, 1)$ -connection \bar{B} on $\tilde{\mathcal{E}}_0$ having zero component along the distribution $V^{0,1}$. In its turn, there is a diffeomorphism between the bundle $(\tilde{\mathcal{E}}_0, \bar{B})$ and the holomorphic vector bundle $\tilde{\mathcal{E}}$ over \mathcal{P} that is trivializable as a smooth bundle over \mathcal{P} and is holomorphically trivializable after restricting to $\mathbb{CP}_x^1 \hookrightarrow \mathcal{P}$, $x \in U$. Thus we have the following equivalence of data:

$$(\mathcal{E}, A) \sim (\tilde{\mathcal{E}}_0, \bar{B}) \sim (\tilde{\mathcal{E}}, \mathcal{F}_{12}),$$

which is called the twistor correspondence between the bundles (\mathcal{E}, A) , $(\tilde{\mathcal{E}}_0, \bar{B})$ and $(\tilde{\mathcal{E}}, \mathcal{F}_{12})$.

3. Čech and Dolbeault descriptions of holomorphic bundles

In the Čech approach holomorphic bundles are described by holomorphic transition matrices, and in the Dolbeault approach they are described by flat $(0,1)$ -connections. In this section we recall definitions of cohomology sets of manifolds with values in sheaves of groups and reformulate the equivalence of the Čech and Dolbeault descriptions of holomorphic bundles in cohomology terms. At last, using the twistor correspondence, we obtain two cohomological descriptions of the moduli space \mathcal{M}_U of self-dual gauge fields.

3.1. Sheaves and cohomology sets

Let us recall some definitions [5, 9]. We consider a complex manifold X , smooth maps from X into a non-Abelian group G and a sheaf \mathfrak{S} of such G -valued functions. Let $\mathfrak{U} = \{\mathcal{U}_\alpha\}$, $\alpha \in I$, be an open covering of the manifold X . A q -cochain of the covering \mathfrak{U} with values in \mathfrak{S} is a collection $\psi = \{\psi_{\alpha_0 \dots \alpha_q}\}$ of sections of the sheaf \mathfrak{S} over nonempty intersections $\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_q}$. A set of q -cochains is denoted by $C^q(\mathfrak{U}, \mathfrak{S})$; it is a group under the pointwise multiplication.

Subsets of cocycles $Z^q(\mathfrak{U}, \mathfrak{S}) \subset C^q(\mathfrak{U}, \mathfrak{S})$ for $q = 0, 1$ are defined as follows

$$Z^0(\mathfrak{U}, \mathfrak{S}) := \{\psi \in C^0(\mathfrak{U}, \mathfrak{S}) : \psi_\alpha \psi_\beta^{-1} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset\}, \quad (12a)$$

$$Z^1(\mathfrak{U}, \mathfrak{S}) := \left\{ \psi \in C^1(\mathfrak{U}, \mathfrak{S}) : \psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset; \right. \\ \left. \psi_{\alpha\beta} \psi_{\beta\gamma} \psi_{\gamma\alpha} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset \right\}. \quad (12b)$$

It follows from (12a) that $Z^0(\mathfrak{U}, \mathfrak{S})$ coincides with the group $H^0(X, \mathfrak{S}) := \mathfrak{S}(X) \equiv \Gamma(X, \mathfrak{S})$ of global sections of the sheaf \mathfrak{S} . The set $Z^1(\mathfrak{U}, \mathfrak{S})$ is not in general a subgroup of the group $C^1(\mathfrak{U}, \mathfrak{S})$.

Cocycles $\hat{f}, f \in Z^1(\mathfrak{U}, \mathfrak{S})$ are called *equivalent* $\hat{f} \sim f$ if $\hat{f}_{\alpha\beta} = \psi_\alpha f_{\alpha\beta} \psi_\beta^{-1}$ for some $\psi \in C^0(\mathfrak{U}, \mathfrak{S})$, $\alpha, \beta \in I$. The cocycle f equivalent to $\hat{f} = 1$ is called *trivial* and for such cocycles $f = \{f_{\alpha\beta}\}$ we have $f_{\alpha\beta} = \psi_\alpha^{-1} \psi_\beta$. A set of equivalence classes of 1-cocycles is called the *1-cohomology set* and denoted by $H^1(\mathfrak{U}, \mathfrak{S})$. After taking the direct limit of the sets $H^1(\mathfrak{U}, \mathfrak{S})$ over successive refinement of the covering \mathfrak{U} of X , one obtains the *Čech 1-cohomology set* $H^1(X, \mathfrak{S})$ of X with coefficients in \mathfrak{S} . In the case when \mathcal{U}_α are Stein manifolds, $H^1(\mathfrak{U}, \mathfrak{S}) = H^1(X, \mathfrak{S})$.

We shall also consider a sheaf $\dot{\mathfrak{S}}$ of smooth functions on X with values in an Abelian group. Then the subgroups of cocycles $Z^q(\mathfrak{U}, \dot{\mathfrak{S}}) \subset C^q(\mathfrak{U}, \dot{\mathfrak{S}})$ for $q = 0, 1$ are defined as follows

$$Z^0(\mathfrak{U}, \dot{\mathfrak{S}}) := \{\theta \in C^0(\mathfrak{U}, \dot{\mathfrak{S}}) : \theta_\alpha - \theta_\beta = 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset\}, \quad (13a)$$

$$Z^1(\mathfrak{U}, \dot{\mathfrak{S}}) := \left\{ \theta \in C^1(\mathfrak{U}, \dot{\mathfrak{S}}) : \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset; \right. \\ \left. \theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \emptyset \right\}, \quad (13b)$$

i.e., everywhere in definitions the multiplication is replaced by addition. Trivial cocycles (coboundaries) are given by the formula $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$, where $\{\theta_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \dot{\mathfrak{S}})$, $\{\theta_\alpha\} \in C^0(\mathfrak{U}, \dot{\mathfrak{S}})$. Quotient spaces (cocycles/coboundaries) are the cohomology spaces $H^i(\mathfrak{U}, \dot{\mathfrak{S}})$, $i = 1, 2, \dots$.

Now we consider the twistor space \mathcal{P} and the two-set open covering $\mathfrak{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ of \mathcal{P} . Then the space of cocycles $Z^1(\mathfrak{U}, \mathfrak{S})$ with coefficients in a sheaf \mathfrak{S} of non-Abelian groups over \mathcal{P} is a special case of formula (12b),

$$Z^1(\mathfrak{U}, \mathfrak{S}) := \{f \in C^1(\mathfrak{U}, \mathfrak{S}) : f_{21} = f_{12}^{-1} \text{ on } \mathcal{U}_1 \cap \mathcal{U}_2\}. \quad (14)$$

Any cocycle $f = \{f_{12}, f_{21}\} \in Z^1(\mathfrak{U}, \mathfrak{S})$ defines a unique complex vector bundle $\tilde{\mathcal{E}}$ over $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ by glueing the direct products $\mathcal{U}_1 \times \mathbb{C}^n$ and $\mathcal{U}_2 \times \mathbb{C}^n$ with the help of G -valued transition matrix f_{12} on \mathcal{U}_{12} . Equivalent cocycles define isomorphic complex vector bundles over \mathcal{P} and smooth complex vector bundles are parametrized by the set $H^1(\mathcal{P}, \mathfrak{S})$.

Let us introduce the sheaf \mathcal{H} of all holomorphic sections of the trivial bundle $\mathcal{P} \times G$, where G is a Lie group. Then holomorphic vector bundles over \mathcal{P} are parametrized by the set $H^1(\mathcal{P}, \mathcal{H})$.

3.2. Exact sequences of sheaves and cohomology sets

We consider the sheaf \mathfrak{S} of smooth sections of the bundle $\mathcal{P} \times G$ and the subsheaf $\mathcal{S} \subset \mathfrak{S}$ of such smooth sections that are annihilated by the distribution $V^{0,1}$ on \mathcal{P} , i.e., locally $\bar{V}_3 \psi = 0$ on $\mathcal{U} \subset \mathcal{P}$. So we have $\mathcal{H} \subset \mathcal{S} \subset \mathfrak{S}$ and there is the canonical embedding $\mathfrak{i} : \mathcal{H} \rightarrow \mathcal{S}$.

Let us also consider the sheaf $\mathcal{B}^{0,1}$ of such smooth $(0,1)$ -forms \bar{B} on \mathcal{P} with values in the Lie algebra \mathcal{G} of G that have zero components along the distribution $V^{0,1}$. Let us define a map $\bar{\delta}^0 : \mathcal{S} \rightarrow \mathcal{B}^{0,1}$ given for any open set $\mathcal{U} \subset \mathcal{P}$ by the formula

$$\bar{\delta}^0 \psi = -(\bar{\partial} \psi) \psi^{-1}, \quad (15)$$

where $\psi \in \mathcal{S}(\mathcal{U})$, $\bar{\delta}^0 \psi \in \mathcal{B}^{0,1}(\mathcal{U})$, $d = \partial + \bar{\partial}$. One can also consider the sheaf $\mathfrak{B}^{0,2}$ of smooth \mathcal{G} -valued $(0,2)$ -forms on \mathcal{P} and introduce an operator $\bar{\delta}^1 : \mathcal{B}^{0,1} \rightarrow \mathfrak{B}^{0,2}$ defined for any open set $\mathcal{U} \subset \mathcal{P}$ by the formula

$$\bar{\delta}^1 \bar{B} = \bar{\partial} \bar{B} + \bar{B} \wedge \bar{B}, \quad (16)$$

where $\bar{B} \in \mathcal{B}^{0,1}(\mathcal{U})$, $\bar{\delta}^1 \bar{B} \in \mathfrak{B}^{0,2}(\mathcal{U})$.

Denote by \mathcal{B} the subsheaf in $\mathcal{B}^{0,1}$ of such \bar{B} that $\bar{\partial} \bar{B} + \bar{B} \wedge \bar{B} = 0$, i.e., $\mathcal{B} = \text{Ker } \bar{\delta}^1$. The sheaf \mathcal{S} acts on the sheaf \mathcal{B} by means of the adjoint representation:

$$\bar{B} \mapsto \text{Ad}_\psi \bar{B} = \psi^{-1} \bar{B} \psi + \psi^{-1} \bar{\partial} \psi.$$

It can be checked that the sequence of sheaves

$$\mathbf{1} \longrightarrow \mathcal{H} \xrightarrow{\mathfrak{i}} \mathcal{S} \xrightarrow{\bar{\delta}^0} \mathcal{B} \xrightarrow{\bar{\delta}^1} 0 \quad (17)$$

is exact, i.e., $\mathcal{B} \simeq \mathcal{S}/\mathcal{H}$. The exact sequence of sheaves induces the following exact sequence of cohomology sets [9, 12]:

$$e \longrightarrow H^0(\mathcal{P}, \mathcal{H}) \xrightarrow{\mathfrak{i}_*} H^0(\mathcal{P}, \mathcal{S}) \xrightarrow{\bar{\delta}_*^0} H^0(\mathcal{P}, \mathcal{B}) \xrightarrow{\bar{\delta}_*^1} H^1(\mathcal{P}, \mathcal{H}) \xrightarrow{\mathfrak{f}} H^1(\mathcal{P}, \mathcal{S}), \quad (18)$$

where e is a marked element of these sets and \mathfrak{f} is an embedding induced by the map \mathfrak{i} .

The sets $H^0(\mathcal{P}, \mathcal{H})$, $H^0(\mathcal{P}, \mathcal{S})$ and $H^0(\mathcal{P}, \mathcal{B})$ are the spaces of global sections of the sheaves \mathcal{H} , \mathcal{S} and \mathcal{B} . The set $H^1(\mathcal{P}, \mathcal{H})$ is the moduli space of holomorphic vector bundles over \mathcal{P} , and the set $H^1(\mathcal{P}, \mathcal{S})$ is the moduli space of smooth complex vector bundles over \mathcal{P} that are holomorphic on any projective line $\mathbb{C}P_x^1 \hookrightarrow \mathcal{P}$, $x \in U$.

3.3. Cohomological description of the moduli space \mathcal{M}_U

By definition the moduli space \mathcal{M}_U of local solutions to the SDYM equations is the space of gauge nonequivalent self-dual connections A on U (see (5)). The space $H^0(\mathcal{P}, \mathcal{B})$ is the space of smooth \mathcal{G} -valued global $(0, 1)$ -forms \bar{B} on \mathcal{P} satisfying (11) and having zero component along the distribution $V^{0,1}$. By virtue of the twistor correspondence $(\mathcal{E}, A) \sim (\tilde{\mathcal{E}}_0, \bar{B})$, the space $H^0(\mathcal{P}, \mathcal{B})$ coincides with the space \mathcal{A}_U of local solutions to the SDYM equations, $H^0(\mathcal{P}, \mathcal{B}) \simeq \mathcal{A}_U$. The group $H^0(\mathcal{P}, \mathcal{S})$ is isomorphic to the group \mathfrak{G}_U of local gauge transformations, because G -valued smooth functions ψ defined globally on $\mathcal{P} = U \times \mathbb{CP}^1$ and holomorphic on \mathbb{CP}^1 do not depend on local complex coordinates of \mathbb{CP}^1 , i.e., $\psi \equiv g(x) \in \mathfrak{G}_U$, $x \in U$. Therefore we have the bijection

$$\mathcal{M}_U \simeq H^0(\mathcal{P}, \mathcal{B})/H^0(\mathcal{P}, \mathcal{S}), \quad (19)$$

that follows from the definition (5) of the moduli space \mathcal{M}_U and the twistor correspondence briefly described in § 2.3. The description of \mathcal{M}_U in terms of \mathcal{G} -valued $(0, 1)$ -forms \bar{B} on \mathcal{P} is called the *Dolbeault description* of \mathcal{M}_U .

Now let us consider the set $\text{Ker } \mathfrak{f} = \mathfrak{f}^{-1}(e)$, $e \in H^1(\mathcal{P}, \mathcal{S})$. It consists of such elements from $H^1(\mathcal{P}, \mathcal{H})$ that are mapped into the class $e \in H^1(\mathcal{P}, \mathcal{S})$ of smoothly trivial complex vector bundles over \mathcal{P} that are holomorphically trivial on any projective line $\mathbb{CP}_x^1 \hookrightarrow \mathcal{P}$, $x \in U$. Therefore, the set $\text{Ker } \mathfrak{f}$ is the moduli space of holomorphic vector bundles $\tilde{\mathcal{E}}$ that are diffeomorphic to the bundle $\tilde{\mathcal{E}}_0$ from the class $e \in H^1(\mathcal{P}, \mathcal{S})$. For any representative $\mathcal{F} = \{\mathcal{F}_{12}, \mathcal{F}_{12}^{-1}\} \in Z^1(\mathfrak{U}, \mathcal{H}) \subset Z^1(\mathfrak{U}, \mathcal{S})$ of the set $\text{Ker } \mathfrak{f}$ one can find a decomposition

$$\mathcal{F}_{12} = \psi_1^{-1}(x, \lambda) \psi_2(x, \lambda^{-1}), \quad (20)$$

where ψ_1, ψ_2 are smooth G -valued functions on $\mathcal{U}_1, \mathcal{U}_2$ that are holomorphic on $\mathbb{CP}_x^1 \hookrightarrow \mathcal{P}$, $x \in U$. Note that $\psi = \{\psi_1, \psi_2\} \in C^0(\mathfrak{U}, \mathcal{S})$.

It follows from the exact sequence (18) that

$$\text{Ker } \mathfrak{f} \simeq H^0(\mathcal{P}, \mathcal{B})/H^0(\mathcal{P}, \mathcal{S}). \quad (21)$$

Therefore we have the bijection

$$\mathcal{M}_U \simeq \text{Ker } \mathfrak{f}, \quad (22)$$

and the description of \mathcal{M}_U in terms of transition matrices $\mathcal{F} \in \text{Ker } \mathfrak{f}$ is called the *Čech description* of the moduli space \mathcal{M}_U .

Let us collect the bijections (19), (21) and (22) in the following table:

the Dolbeault description	the moduli space of self-dual gauge fields	the Čech description
$H_{\bar{\partial}_B}^{0,1}(\mathcal{P}) \supset H^0(\mathcal{P}, \mathcal{B})/H^0(\mathcal{P}, \mathcal{S})$	$\simeq \mathcal{M}_U \simeq$	$\text{Ker } \mathfrak{f} \subset H^1(\mathcal{P}, \mathcal{H}),$

where $H_{\bar{\partial}_B}^{0,1}(\mathcal{P})$ is a Dolbeault 1-cohomology set defined as a set of orbits of the group $H^0(\mathcal{P}, \mathfrak{S})$ in the set $H^0(\mathcal{P}, \mathfrak{B})$ and \mathfrak{B} is the sheaf of \mathcal{G} -valued $(0, 1)$ -forms \hat{B} on \mathcal{P} such that $\bar{\partial}_B^2 = 0$.

4. Infinitesimal symmetries of the SDYM equations

We can now use the results of the previous sections to study symmetries of the SDYM equations. Cohomological description of the moduli space of self-dual gauge fields simplifies the problem of finding symmetries of the SDYM equations and clarifies the geometric meaning of these symmetries. Namely, in the Čech approach, to solutions of the SDYM equations there correspond holomorphic G -valued functions \mathcal{F}_{12} (1-cocycles) on the overlap \mathcal{U}_{12} of the open sets $\mathcal{U}_1, \mathcal{U}_2$ covering the twistor space \mathcal{P} . Therefore any holomorphic perturbation of \mathcal{F}_{12} determines a tangent vector on the solution space of the SDYM equations. In § 4.2 we define these infinitesimal holomorphic transformations of \mathcal{F}_{12} by multiplying \mathcal{F}_{12} on holomorphic \mathcal{G} -valued matrices θ_{12}, θ_{21} defined on \mathcal{U}_{12} . Then, using a solution of the infinitesimal variant of the Riemann-Hilbert problem from § 4.3, we proceed in § 4.4 to the Dolbeault description and define a transformation of the flat $(0,1)$ -connection \bar{B} . Finally we introduce the algebra $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ of 1-cochains of \mathcal{P} with values in the sheaf $\dot{\mathcal{H}}$ of \mathcal{G} -valued holomorphic functions on \mathcal{P} and, using the Penrose-Ward correspondence, we describe in § 4.5 the action of the algebra $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ on self-dual gauge potentials.

4.1. Action of the group $C^1(\mathfrak{U}, \mathcal{H})$ on the space $Z^1(\mathfrak{U}, \mathcal{H})$

The group $C^1(\mathfrak{U}, \mathcal{H})$ and the space $Z^1(\mathfrak{U}, \mathcal{H})$ have been described in § 3.1. Let us define the action ρ of $C^1(\mathfrak{U}, \mathcal{H})$ on $Z^1(\mathfrak{U}, \mathcal{H})$ by the formula

$$(\rho_h f)_{12} = h_{12} f_{12} h_{21}^{-1}, \quad (23)$$

where $h = \{h_{12}, h_{21}\} \in C^1(\mathfrak{U}, \mathcal{H})$, $f = \{f_{12}, f_{12}^{-1}\} \in Z^1(\mathfrak{U}, \mathcal{H})$. It is clear that for an arbitrary cocycle $f = \{f_{12}, f_{21}\} \in Z^1(\mathfrak{U}, \mathcal{H})$, one can always find a cochain $\{h_{12}, h_{21}\} \in C^1(\mathfrak{U}, \mathcal{H})$ such that $f_{12} = h_{12} h_{21}^{-1}$, $f_{21} = h_{21} h_{12}^{-1}$, i.e., the group $C^1(\mathfrak{U}, \mathcal{H})$ acts transitively on $Z^1(\mathfrak{U}, \mathcal{H})$. The stability subgroup of the trivial cocycle $f^0 = 1$ is

$$C_\Delta(\mathfrak{U}, \mathcal{H}) = \{\{h_{12}, h_{21}\} \in C^1(\mathfrak{U}, \mathcal{H}) : h_{12} = h_{21}\}.$$

Therefore, $Z^1(\mathfrak{U}, \mathcal{H})$ is a homogeneous space,

$$Z^1(\mathfrak{U}, \mathcal{H}) = C^1(\mathfrak{U}, \mathcal{H}) / C_\Delta(\mathfrak{U}, \mathcal{H}).$$

4.2. Action of the algebra $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ on the space $Z^1(\mathfrak{U}, \mathcal{H})$

Let us denote by $\dot{\mathcal{H}}$ the sheaf of holomorphic sections of the trivial bundle $\mathcal{P} \times \mathcal{G}$, where \mathcal{G} is the Lie algebra of a Lie group G . Denote by $\dot{\mathcal{S}}$ the sheaf of smooth partially holomorphic sections of the bundle $\mathcal{P} \times \mathcal{G}$, i.e., such smooth maps $\phi : \mathcal{P} \rightarrow \mathcal{G}$ that $\partial_{\bar{\lambda}} \phi = 0$ in the local coordinates $\{x^\mu, \lambda, \bar{\lambda}\}$ on \mathcal{P} .

We consider the infinitesimal form of the action (23). Substituting $h_{12} = \exp(\theta_{12}) \simeq 1 + \theta_{12}$, $h_{21} = \exp(\theta_{21}) \simeq 1 + \theta_{21}$, we have

$$\delta_\theta \mathcal{F}_{12} = \theta_{12} \mathcal{F}_{12} - \mathcal{F}_{12} \theta_{21}, \quad (24)$$

where $\theta = \{\theta_{12}, \theta_{21}\} \in C^1(\mathfrak{U}, \dot{\mathcal{H}})$, $\mathcal{F} = \{\mathcal{F}_{12}, \mathcal{F}_{12}^{-1}\} \in Z^1(\mathfrak{U}, \mathcal{H})$. Here and in what follows as $\mathcal{F} = \{\mathcal{F}_{12}, \mathcal{F}_{12}^{-1}\}$ we take representatives of the space $\text{Ker } f$ (see § 3.3), i.e., such cocycles \mathcal{F}_{12} that admits the decomposition (20).

4.3. The map $\phi : C^1(\mathfrak{U}, \dot{\mathcal{H}}) \rightarrow C^0(\mathfrak{U}, \dot{\mathcal{S}})$

Now we construct the following \mathcal{G} -valued function:

$$\Phi_{12}(\theta) = \psi_1(\delta_\theta \mathcal{F}_{12}) \psi_2^{-1}, \quad (25)$$

where $\{\psi_1, \psi_2\} \in C^0(\mathfrak{U}, \mathcal{S})$ and $\mathcal{F}_{12} = \psi_1^{-1} \psi_2$. Then one can check that

$$\Phi_{21} = -\Phi_{12}$$

and Φ_{12} is a smooth \mathcal{G} -valued function on \mathcal{U}_{12} such that $\partial_{\bar{\lambda}} \Phi_{12} = 0$ in the local coordinates $\{x^\mu, \lambda, \bar{\lambda}\}$ on \mathcal{U}_{12} . Therefore, $\Phi = \{\Phi_{12}, \Phi_{21}\} \in Z^1(\mathfrak{U}, \dot{\mathcal{S}})$.

It can be shown that $H^1(\mathcal{P}, \dot{\mathcal{S}}) = 0$, since $\dot{\mathcal{S}}$ is the sheaf of smooth \mathcal{G} -valued functions on \mathcal{P} that are holomorphic on $\mathbb{C}P^1 \hookrightarrow \mathcal{P}$. Therefore, each 1-cocycle with values in $\dot{\mathcal{S}}$ is a 1-coboundary, and we have

$$\Phi_{12}(\theta) = \phi_1(\theta) - \phi_2(\theta), \quad (26)$$

where $\phi(\theta) = \{\phi_1(\theta), \phi_2(\theta)\} \in C^0(\mathfrak{U}, \dot{\mathcal{S}})$.

Notice that the splitting (26) defined for any $\theta \in C^1(\mathfrak{U}, \dot{\mathcal{H}})$ is not unique. Namely, as a 0-cochain from $C^0(\mathfrak{U}, \dot{\mathcal{S}})$ instead of $\phi(\theta)$ one can also take

$$\tilde{\phi}(\theta) = \{\phi_1(\theta) + \varphi_1, \phi_2(\theta) + \varphi_2\},$$

where $\varphi_1 = \varphi_2$ on \mathcal{U}_{12} , i.e., $\varphi = \{\varphi_1, \varphi_2\} \in H^0(\mathcal{P}, \dot{\mathcal{S}})$. Let us fix $\varphi \in H^0(\mathcal{P}, \dot{\mathcal{S}})$ for each $\theta \in C^1(\mathfrak{U}, \dot{\mathcal{H}})$, then the splitting (26) defines a subspace $\phi(C^1(\mathfrak{U}, \dot{\mathcal{H}}))$ in $C^0(\mathfrak{U}, \dot{\mathcal{S}})$. It can be checked that

$$\phi([\theta, \tilde{\theta}]) = [\phi(\theta), \phi(\tilde{\theta})] = \{[\phi_1(\theta), \phi_1(\tilde{\theta})], [\phi_2(\theta), \phi_2(\tilde{\theta})]\} \in C^0(\mathfrak{U}, \dot{\mathcal{S}})$$

for any $\theta, \tilde{\theta} \in C^1(\mathfrak{U}, \dot{\mathcal{H}})$. Therefore, the map $\phi : C^1(\mathfrak{U}, \dot{\mathcal{H}}) \rightarrow C^0(\mathfrak{U}, \dot{\mathcal{S}})$ is a homomorphism.

4.4. Action of the algebra $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ on the space $H^0(\mathcal{P}, \mathcal{B})$

Using the action (24) and the homomorphism ϕ , we obtain an action

$$\delta_\theta \psi_1 = -\phi_1(\theta) \psi_1, \quad \delta_\theta \psi_2 = -\phi_2(\theta) \psi_2, \quad (27)$$

of the algebra $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ on a 0-cochain $\{\psi_1, \psi_2\} \in C^0(\mathfrak{U}, \mathcal{S})$ such that $\mathcal{F}_{12} = \psi_1^{-1} \psi_2$.

By definition, for $\bar{B} = \{\bar{B}^{(1)}, \bar{B}^{(2)}\} \in H^0(\mathcal{P}, \mathcal{B})$ we have

$$\bar{B}^{(1)} = -(\bar{\partial} \psi_1) \psi_1^{-1} \text{ on } \mathcal{U}_1, \quad \bar{B}^{(2)} = -(\bar{\partial} \psi_2) \psi_2^{-1} \text{ on } \mathcal{U}_2, \quad \bar{B}^{(1)} = \bar{B}^{(2)} \text{ on } \mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2.$$

Therefore, the action of $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ on $H^0(\mathcal{P}, \mathcal{B})$ has the form

$$\delta_\theta \bar{B}^{(1)} = \bar{\partial} \phi_1(\theta) + [\bar{B}^{(1)}, \phi_1(\theta)], \quad (28a)$$

$$\delta_\theta \bar{B}^{(2)} = \bar{\partial} \phi_2(\theta) + [\bar{B}^{(2)}, \phi_2(\theta)]. \quad (28b)$$

The transformations (28) look like infinitesimal gauge transformations

$$\delta_\varphi \bar{B} = \bar{\partial} \varphi + [\bar{B}, \varphi], \quad (29)$$

where φ is an element of the Lie algebra $H^0(\mathcal{P}, \dot{\mathcal{S}}) \simeq \mathfrak{g}_U$ of the gauge group $H^0(\mathcal{P}, \mathcal{S}) \simeq \mathfrak{G}_U$. But for $\phi(\theta) = \{\phi_1(\theta), \phi_2(\theta)\} \in C^0(\mathfrak{U}, \dot{\mathcal{S}})$ we have $\phi_1(\theta) \neq \phi_2(\theta)$ on \mathcal{U}_{12} , and the transformations (28) differ from (29).

4.5. Action of the algebra $C^1(\mathfrak{U}, \mathcal{H})$ on the space \mathcal{A}_U

Recall that we consider a self-dual 4-manifold M , the twistor space \mathcal{Z} of which is a complex 3-manifold, and the SDYM equations (1) on M . To describe infinitesimal symmetries of the SDYM equations, we take an open ball $U \subset M$ and the twistor space \mathcal{P} of U that is covered by two coordinate patches \mathcal{U}_1 and \mathcal{U}_2 (see § 2.2).

The twistor correspondence gives us the following relation between a self-dual connection $A = A_\mu dx^\mu$ on the complex vector bundle \mathcal{E} over U and a flat $(0, 1)$ -connection $\bar{B} = \{\bar{B}^{(1)}, \bar{B}^{(2)}\}$ on the bundle $\tilde{\mathcal{E}}_0 = \pi^* \mathcal{E}$:

$$\bar{B}_1^{(1)} = A_{\bar{y}} - \lambda A_z, \quad \bar{B}_2^{(1)} = A_{\bar{z}} + \lambda A_y, \quad \bar{B}_3^{(1)} = 0 \text{ on } \mathcal{U}_1, \quad (30a)$$

$$\bar{B}_1^{(2)} = \zeta A_{\bar{y}} - A_z, \quad \bar{B}_2^{(2)} = \zeta A_{\bar{z}} + A_y, \quad \bar{B}_3^{(2)} = 0 \text{ on } \mathcal{U}_2, \quad (30b)$$

where $y = x^1 + ix^2$, $z = x^3 - ix^4$, $\bar{y} = x^1 - ix^2$, $\bar{z} = x^3 + ix^4$ are complex coordinates on U .

One can always choose such local frames $\{\bar{V}_a^{(1)}\}$, $\{\bar{V}_a^{(2)}\}$ of the bundle $T^{0,1}$ over \mathcal{U}_1 , \mathcal{U}_2 , respectively, that $[\bar{V}_a^{(1)}, \bar{V}_b^{(1)}] = 0$, $[\bar{V}_a^{(2)}, \bar{V}_b^{(2)}] = 0$, $\bar{V}_3^{(1)} = \partial_{\bar{\lambda}}$, $\bar{V}_3^{(2)} = \partial_{\bar{\zeta}}$ and on the intersection $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$ the local frames are connected by the formulae [10, 1, 15]

$$\bar{V}_1^{(1)} = \lambda \bar{V}_1^{(2)}, \quad \bar{V}_2^{(1)} = \lambda \bar{V}_2^{(2)}, \quad \bar{V}_3^{(1)} = -\bar{\lambda}^2 \bar{V}_3^{(2)}.$$

From (27), (28) we obtain the following action of the algebra $C^1(\mathfrak{U}, \mathcal{H})$ on the space \mathcal{A}_U of solutions to the SDYM equations on U :

$$\begin{aligned} \delta_\theta A_y &= \oint_{S^1} \frac{d\lambda}{2\pi i \lambda} (\bar{V}_2^{(2)} + \bar{B}_2^{(2)}) \phi_2(\theta), & \delta_\theta A_z &= - \oint_{S^1} \frac{d\lambda}{2\pi i \lambda} (\bar{V}_1^{(2)} + \bar{B}_1^{(2)}) \phi_2(\theta), \\ \delta_\theta A_{\bar{y}} &= \oint_{S^1} \frac{d\lambda}{2\pi i \lambda} (\bar{V}_1^{(1)} + \bar{B}_1^{(1)}) \phi_1(\theta), & \delta_\theta A_{\bar{z}} &= \oint_{S^1} \frac{d\lambda}{2\pi i \lambda} (\bar{V}_2^{(1)} + \bar{B}_2^{(1)}) \phi_1(\theta), \end{aligned} \quad (31)$$

where $S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\}$.

5. Conclusion

The space of local solutions to the SDYM equations on a self-dual 4-manifold M has been considered. Choosing the concrete self-dual 4-manifold (e.g. S^4 , T^4 , ...) or imposing some boundary conditions on gauge fields, one can obtain instantons, monopoles or other special solutions of the SDYM equations, the moduli spaces of which are discussed in the talk by S.T.Tsou [13]. Our purpose was to describe the moduli space and symmetries of *local* solutions to the SDYM equations. The use of twistor correspondence and cohomologies reveals the geometric meaning of symmetries of the SDYM equations, which may help in quantizing the SDYM model.

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